Proceedings of Symposia in Pure Mathematics
Volume 32, 1978

RECENT RESULTS ON SUFFICIENTLY LARGE 3-MANIFOLDS

FRIEDHELM WALDHAUSEN

This is an expository paper, an expanded version of the talk actually given (which went only to what is §2 of this paper). The topics discussed are:

Johannson's *classification of exotic homotopy equivalences*;

Hemion's *classification of homeomorphisms of a 2-manifold* (compact, with nonempty boundary).

It will be indicated that (and in what sense) some of the main problems on sufficiently large irreducible 3-manifolds can now be considered solved: Classification, classification up to homotopy type, classification of manifolds homotopy equivalent to a given one, classification of knots, classification of knot groups.

The plan of the paper is as follows.

§1 gives background material on exotic homotopy equivalences and in particular some examples.

§2 introduces the *characteristic submanifold*; this notion is needed in the statement of Johannson's result. The result is then discussed.

§3 introduces *manifolds with boundary pattern*, a relativization of 3-manifolds required for inductive proofs. A rough indication of proof of Johannson's result is included.

§4 discusses Haken's approach to classification. The language of the preceding sections is used (at least part of this was indeed implicitly used by Haken). It is indicated how Hemion's result provides the missing step in Haken's theory. Some related results are also discussed.

1. Prelude to homotopy equivalences. The question is: If $f : M \to N$ is a homotopy equivalence of 3-manifolds, what conditions guarantee that f is homotopic to a homeomorphism?

One sufficient set of conditions is the following [24] (everything PL, say).

AMS (MOS) subject classifications (1970). Primary 55D10, 57A05, 57A10; Secondary 57C25.

(1) M should be compact, orientable, and irreducible (that is, every 2-sphere in M bounds a 3-ball in M).

(2) If M is closed it should be sufficiently large (that is, there should exist an embedding of a closed orientable 2-manifold in M whose fundamental group is nontrivial and injects).

(3) If the boundary $\partial M \neq \emptyset$ the map f should actually be given as a homotopy equivalence of pairs $M, \partial M \to N, \partial N$.

The status of these conditions is, roughly, the following: 'compact' and 'orientable' are mainly asked for convenience. That is, each can be replaced by a considerably weaker but more technical condition, and everything goes through without essential change. For example in the nonorientable case one may simply define the projective planes away; and in the noncompact case one may insist on maps being proper and manifolds being 'sufficiently large at infinity' (one way to put the latter is to ask that the pro-object of fundamental groupoids at infinity be isomorphic to one in which all maps of vertex groups are injective).

'Irreducibility' is justified by the Kneser-Haken-Milnor unique decomposition theorem. It is of great importance, both technically and otherwise. Its purpose is twofold. It serves to get around the unresolved Poincaré conjecture, and it serves to avoid splitting problems at 2-spheres. The latter have actually been solved by Laudenbach, Swarup, and, ultimately, Hendriks.

The condition 'sufficiently large' is being discussed elsewhere **[27]**. Notice it is only asked for closed manifolds.

We are here interested in the problem of omitting the condition prescribing f on the boundary.

EXAMPLE 1.1. Suppose there is a component G of ∂M so that $\ker(\pi_1 G \to \pi_1 M) \neq 0$. By the loop theorem of Papakyriakopoulos, cf. **[20]**, this means that we can write $M = M' \cup$ 1-handle attached at $\partial M'$, and it is obvious that there are many homotopy equivalences from M to itself which map this 1-handle through the interior. The situation is messy but comparatively easy to analyse. For example the homotopy equivalences fixing M' are given by group theory.

To avoid this phenomenon we consider henceforth only 3-manifolds whose boundary is (nonempty and) incompressible, that is, for any component G of ∂M, $\pi_1 G \to \pi_1 M$ is injective. We also insist on condition (1) above. The class of manifolds still being considered includes some of the most interesting 3-manifolds, in particular it includes the knot spaces of nontrivial knots.

For this class of manifolds one knows **[24]**: If $f : M \to N$ is a homotopy equivalence, and if there exists f' homotopic to f with $f'(\partial M) \subset \partial N$, then f is homotopic to a homeomorphism.

So we have focussed our attention, for the class of manifolds being considered, on

Problem. Suppose $f: M \to N$ is a homotopy equivalence. Suppose there does not exist f' homotopic to f with $f'(\partial M) \subset \partial N$. What can one say?

We will call such homotopy equivalences *exotic*. Here are some examples of exotic homotopy equivalences.

EXAMPLE 1.2. Let F_1 and F_2 be, respectively, the 2-torus with one open disk removed and the 2-sphere with three open disks removed. There is a homotopy equivalence $F_1 \to F_2$. Then $F_1 \times S^1 \to F_2 \times S^1$ is exotic.

EXAMPLE 1.3. Assume given M_1, M_2 and embeddings of the annulus, f_i: $S^1 \times I \to \partial M_i$ with f_{i*} injective on π_1. Let h: $S^1 \times I \to S^1 \times I$ be given by the flip of the interval. Form M (resp., N) by gluing M_1 and M_2 at $S^1 \times I$ by means of f_1 and f_2 (resp., $f_1 \circ h$ and f_2). Then M and N are homotopy equivalent but in general not homeomorphic.

Note that in Example 1.2 the number of boundary components changes. In Example 1.3 their type may change as well.

EXAMPLE 1.4. Let M have a Seifert fibre space structure with decomposition surface B (and $\partial B \neq \emptyset$ since $\partial M \neq \emptyset$), or to use current language, M has a (stable) foliation by circles, and B is the space of leaves. Then M may admit exotic (self-) homotopy equivalences of the type of Example 1.2, i.e., exotic homotopy equivalences which are induced from 2-manifold phenomena. But there may also be other ones. Specifically, the foliation is characterized by B and the nontrivial monodromy; this occurs at isolated singular leaves, say r in number, and in each case may be specified by a coprime pair (α_i, β_i), $0 < \beta_i < \alpha_i$. Then, except for the cases (B = 2-disk and $r \leq 1$) and (B = Möbius strip and $r = 0$), the type of B and the set of the (α_i, β_i) are an invariant of the oriented homeomorphism type of M **[22]**. But the homotopy type is only given by the homotopy type of B and the set of the α_i (no β_i) **[23]**.

The additional phenomena in Example 1.4 may be traced to the following

EXAMPLE 1.5. Let $M = S^1 \times D^2$ be the solid torus, and f: $S^1 \times I \to \partial M$ an embedding of the annulus, with winding number $\alpha > 1$. Letting $F = f(S^1 \times I)$, the oriented homeomorphism type of (M, F) is characterized (among such pairs) by an integer β, coprime with α, and $0 < \beta < \alpha$. But the homotopy type of the pair (M, F) is characterized by α alone.

This last example does not really fit into the framework we have been considering so far. It just illustrates that 'relative' phenomena may manifest themselves in a nonrelativized framework.

The reader may amuse himself in looking for more examples of exotic homotopy equivalences. Leaving aside modifications of the examples given, he will probably find the search rather difficult—with reason, as we shall see later.

All of the examples given have one thing in common. Underlying any of them is a very simple, and very special, geometric phenomenon. It will turn out there is a system. One may compare the situation to the ancient myth of the stable of one Augias. In that tale, after a considerable effort to dispose of the obvious, what remained was much cleaner.

2. Classification of homotopy equivalences. The results to be described are due to Johannson **[9]**, **[10]**, **[11]** and partly myself **[26]**; partial results have been rediscovered by Feustel, Jaco, Shalen, and others, in a large number of papers.

To begin with, one considers a special case. Suppose f: $M \to N$ is an exotic homotopy equivalence, and each component of ∂M is a torus. Then by definition of the terms involved there exists in N an *essential singular torus* which cannot be deformed into the boundary, in the following sense.

DEFINITION. g: $S^1 \times S^1 \to N$ is *essential* if g_* is injective on π_1.

This draws attention to essential singular tori. One would like to analyse them à la loop theorem and sphere theorem. However the cut and paste technique turns out to be inadequate, mainly because the naive conjecture has easy counterexamples.

EXAMPLE 2.1. In the knot space of a torus knot, any nonsingular essential torus is parallel to the boundary [22].

EXAMPLE 2.2. In any of the Seifert fibre spaces which furnish the known examples of nonsufficiently large 3-manifolds (closed, irreducible, with infinite fundamental group) there is an essential singular torus, in fact there are infinitely many such, but there is no incompressible surface whatsoever [23]. Also there is an infinite number of manifolds M which are Seifert fibre spaces over S^2 with precisely three exceptional fibres, and which satisfy $H^1(M) \neq 0$ and are hence sufficiently large. Any such M contains a unique incompressible surface, up to isotopy (it is a fibre in some fibration over S^1), but for only finitely many M can this surface be a torus [22].

Turning hindsight into foresight one decides to consider *all* essential tori, singular or not, all at once, hoping to force them into a pattern. Having decided this far, it is clearly unreasonable not to consider at the same time essential annuli, singular or not.

DEFINITION. $g\colon (S^1 \times I, S^1 \times \partial I) \to (N, \partial N)$ is *essential* if

$$g_*\colon \pi_1(S^1 \times I) \to \pi_1 N, \qquad g_*\colon \pi_1(S^1 \times I, \partial) \to \pi_1(N, \partial)$$

are both injective.

One considers now submanifolds of a given manifold M which in a sense can be manufactured out of essential tori and annuli. It is convenient to change here our conventions about M. In addition to the manifolds considered up to this point we also admit manifolds that are closed (orientable and irreducible) and sufficiently large.

DEFINITION 2.3. A compact codimension zero submanifold V of M is an *essential F-manifold* ('F' for 'fibering') if and only if for each component W of V at least one of the following holds: either

(a) (i) W admits a structure of Seifert fibre space, $p\colon W \to B$, such that $p^{-1}(p(W \cap \partial M)) = W \cap \partial M$, and

(ii) each component of $\mathrm{Cl}(\partial W - \partial M)$ is an essential annulus or torus, or

(b) (i) W admits a structure of line bundle, $p\colon W \to B$ such that $W \cap \partial M$ is the associated 0-sphere bundle, and

(ii) each component of $\mathrm{Cl}(\partial W - \partial M)$ is an essential annulus.

DEFINITION 2.4. A *characteristic submanifold* of M is an essential F-manifold V in M satisfying

(i) if X is any essential F-manifold in M then X can be properly isotoped into V,

(ii) if Y is any union of components of $\mathrm{Cl}(M - V)$ then $V \cup Y$ is not an essential F-manifold.

A condition equivalent to (ii) is that one cannot throw away a component of V and still have (i). Thus V is definable by a universal property.

THEOREM 1. *The characteristic submanifold of M exists and is unique up to ambient isotopy.*

The status of the theorem is this. Once conceived of, the definition of characteristic submanifold may easily be reformulated to involve some kind of 'complexity'. Existence is then provable by the method of the Kneser-Haken finiteness theorem [5]. Given the existence, uniqueness is not hard to show.

EXAMPLE 2.5. With the present (nonrelativized) definition of the characteristic submanifold one has $V = M$ if and only if M is either a Seifert fibre space or a line bundle over a closed 2-manifold.

EXAMPLE 2.6. Let M be a *graph manifold* in the sense of [22]. Then, in general, the following is true: There exists a system T of incompressible tori, unique up to isotopy, with the following properties: (i) if $U(T)$ denotes a regular neighborhood then each component of $\mathrm{Cl}(M - U(T))$ admits the structure of a Seifert fibre space, (ii) no subsystem of T has property (i). In this case $V = \mathrm{Cl}(M - U(T))$. It is disputable if one should not rather adjust the definition of characteristic submanifold so that $V = M$ in this situation. However the smaller V the better are the results one formulates using it. The thing to remember from this example is the following. If one removes V from M, i.e., forms $\mathrm{Cl}(M - V)$, then there may be some 'trivial components' left over, such as $U(T)$ in the example.

THEOREM 2. *Essential singular annuli and tori can be deformed into the characteristic submanifold.*

For example suppose there exists at least one essential singular torus in M. Then the characteristic submanifold V of M cannot be empty, by the theorem. In general one will have $V \neq M$, thus $\mathrm{Cl}(\partial V - \partial M) \neq \emptyset$. But by definition of V, any component of $\mathrm{Cl}(\partial V - \partial M)$ is an essential torus or annulus. Therefore such must exist in M. In the special case when M equals its characteristic submanifold, it is a Seifert fibre space or a line bundle. In Seifert fibre spaces, in general, essential tori do exist in large numbers. But in very special cases there may be none at all, cf. Example 2.2.

The slogan is that it takes many nonsingular annuli or tori to manufacture one singular one. Also one manufactures them in a very special way and still gets them all. That is, essential singular annuli and tori in Seifert fibre spaces and line bundles can be fairly explicitly classified, and in particular any such map can be deformed into the composition of a covering map and an immersion without triple points.

Working in a suitable relative framework, and using the notion of essential map, both of which will be discussed later, one may formulate a corollary giving a version of the theorem for maps of Seifert fibre spaces and line bundles. One way to put the corollary is to say that the universal property of the characteristic submanifold continues to hold for 'singular essential F-manifolds'. In a very special case this amounts to the following: If M has a finite covering which is a Seifert fibre space then M must be a Seifert fibre space itself.

To formulate the main theorem it is convenient to make the following definition.

DEFINITION. Let $f: M \to N$ be a homotopy equivalence, and $M' \subset M$. One says f has *singular support in* M' if and only if there exist $N' \subset N$ and $f': M \to N$ homotopic to f with the following properties

(i) $f'(M') \subset N'$,

(ii) $f'(M - M') \subset N - N'$,

(iii) $f'|M': M' \to N'$ is a homotopy equivalence,

(iv) $f'|\mathrm{Cl}(M - M'): \mathrm{Cl}(M - M') \to \mathrm{Cl}(N - N')$ is a homeomorphism.

Let M, N be as specified earlier.

THEOREM 3. *Every homotopy equivalence $f: M \to N$ has singular support in the characteristic submanifold of M.*

The theorem admits an immediate strengthening. Namely if W is a component of V, the characteristic submanifold, and W is in the interior of M, then for the f' above it is true that $f'|W$ is a boundary preserving homotopy equivalence, hence deformable into a homeomorphism. Hence

COROLLARY. *Every homotopy equivalence has singular support in V', the union of those components of V which meet the boundary of M.*

Thus all exotic homotopy equivalences are just modifications of the Examples 1.2, 1.3, 1.4 described earlier.

A special case is when V' is 'trivial', that is, contained in a neighborhood of ∂M (there could be boundary tori). Obviously this is the case if and only if there is no essential annulus in M.

COROLLARY. *No essential annulus, no exotic homotopy equivalence.*

EXAMPLE. In the knot space of a nontrivial knot there exists an essential annulus if and only if the knot is either a composite knot or a cable or torus knot, respectively.

Conversely, in these special cases, it is in general easy to exhibit exotic homotopy equivalences. One may expect to do better if one restricts attention to homotopy equivalences *between knot spaces.* This is clear in the case of torus knots. In the case of cable knots the matter depends on the unresolved status of the unique embedding conjecture, that any embedding of a nontrivial knot space into S^3, can be extended to an automorphism of S^3. In writing the announcement **[26]** I was under the impression that one could get around the unique embedding conjecture by a trick, but in fact one cannot as was shown to me years ago by an explicit construction of hypothetical counterexamples, by John Hempel (the examples were not published then, they have subsequently been rediscovered by J. Simon). At any rate it is not difficult to prove the following; notice the unique embedding conjecture is used twice.

COROLLARY. *If it is true that nontrivial knot spaces have the unique embedding property then noncomposite knots are characterized by their groups.*

Here are some comments on the status of Theorem 3. Given Theorem 2, the proof of Theorem 3 is fairly easy in the case of manifolds whose boundary consists of tori only. The point is that in this case ∂M is actually contained in the characteristic submanifold, so special arguments apply. Unfortunately nothing like this is true in the general case. Indeed the proof of Theorem 3 is quite complicated in the general case.

The main trick by which one makes Theorem 3 provable at all is to formulate a more general assertion. This involves the relative framework of 'manifolds with boundary pattern'. A bonus is that the proof of Theorem 2 becomes relatively easy, in that framework.

3. Manifolds with boundary, revisited. Let M be a compact n-manifold, $n \leq 3$. A *boundary pattern* for M consists of a set of compact connected $(n-1)$-manifolds in the boundary ∂M which meet nicely, that is, the intersection of any two of them is an $(n-2)$-manifold, the intersection of any three is an $(n-3)$-manifold, and so on.

The boundary pattern $\{F_i\}$ of M is *complete* if $\bigcup F_i = \partial M$. In general one may define the *completed boundary pattern* to consist of $\{F_i\}$ plus the set of connected components of $\mathrm{Cl}(\partial M - \bigcup F_i)$.

A *map* from $(M, \{F_i\})$ to $(N, \{G_j\})$ is a pair of maps $f\colon M \to N$, $\nu\colon \{F_i\} \to \{G_j\}$ such that $f(F_i) \subset G_{\nu(i)}$ and such that

$$\{F_i\} = \bigcup_j \text{(set of connected components of } f^{-1}(G_j) \cap \partial M).$$

A loose way to phrase this is to say that $\{F_i\}$ must be induced, by means of f, from the boundary pattern $\{G_j\}$. In particular it is never possible that nondisjoint members of $\{F_i\}$ are mapped to the same G_j.

A *homotopy* is a continuous family of maps, in the sense just defined, satisfying that the map of index sets does not change. Having defined 'homotopy' one also has defined 'isotopy', 'homotopy equivalence', and so on.

The unit interval is canonically a 1-manifold with complete boundary pattern. A *singular arc* in $(M, \{F_i\})$ is, by definition, a map $f\colon (I, \{0, 1\}) \to (M, \{F_i\})$. It is called *inessential* if f can be deformed to a point map (note this may happen even if $f(0)$ and $f(1)$ are in distinct, but adjacent, elements of $\{F_i\}$), otherwise it is called essential. An essential singular curve is a map $f\colon S^1 \to M$ that cannot be deformed to a point map. Using these notions we may define a map $(M, \{F_i\}) \to (N, \{G_j\})$ to be *essential* if it preserves essential curves and arcs.

DEFINITION 3.1. A boundary pattern $\{F_i\}$ of M is *useful* if and only if for any j the embedding

$$\Big(F_j, \bigcup_{i \neq j} \{\text{connected components of } F_i \cap F_j\}\Big) \longrightarrow (M, \{F_i\})$$

is an essential map.

This is the proper notion to work with, whence the name.

EXAMPLE. Let *i-faced disk* denote a 2-disk with complete boundary pattern of i elements. This is a 2-manifold with useful boundary pattern only if $i \geqq 4$. A 4-faced disk will be referred to as a *square*; this is isomorphic to $I \times I$ as a manifold with boundary pattern.

REMARK. Call an embedding of an i-faced disk, $i \leqq 3$, in $(M, \{F_i\})$ 'uninteresting' if it is isotopic to an embedding into ∂M so that $D \cap \bigcup \partial F_i$ is isomorphic to the cone on $\partial D \cap \bigcup \partial F_i$. There is a version of the loop theorem for 3-manifolds with boundary pattern (it is more or less equivalent to the main technical result of **[25]**). It says that (except for a few degenerate cases) usefulness is equivalent to the nonexistence of interesting i-faced disks, $i \leqq 3$.

EXAMPLE. Let $M = S^1 \times D^2$, the solid torus. Let $F \subset \partial M$ be an annulus, with winding number w (that is, $\pi_1 F$ has index w in $\pi_1 M$). Then the completed boundary pattern of $\{F\}$ is useful if and only if $w \geqq 2$.

REMARK. Still following our earlier convention that the 3-manifolds under consideration are compact, orientable, irreducible, and sufficiently large, let $f\colon (M, \{F_i\}) \to (N, \{G_j\})$ be a homotopy equivalence of 3-manifolds with boundary patterns which are both complete and useful. Then f is homotopic to a homeomorphism. This can be seen by an adaptation of the argument of **[24]**. Indeed the adaptation clarifies the argument.

We will now adapt the notion of essential F-manifold to 3-manifolds with boundary pattern. To avoid introducing even more language this will be done only in the case of complete boundary patterns.

DEFINITION 3.2. Let M be a 3-manifold with boundary pattern $\{F_i\}$, both complete and useful. An *essential F-manifold* in $(M, \{F_i\})$ is an embedded 3-manifold with boundary pattern

$$\left(V, \bigcup_i \{\text{connected components of } V \cap F_i\}\right)$$

whose completed boundary pattern is useful. Furthermore for each component W of V, and its induced boundary pattern, at least one of the following must be true; either

(a) W admits a structure of Seifert fibre space, with fibre projection $p\colon W \to B$, such that the boundary pattern of W is induced, by means of p, from some boundary pattern of B; or

(b) W admits a structure of line bundle, $p\colon W \to B$, such that the boundary pattern of W consists of the components of the associated 0-sphere bundle, plus the boundary pattern induced, by means of p, from some boundary pattern of B.

One defines a *characteristic submanifold* of $(M, \{F_i\})$ as an essential F-manifold having a certain universal property, just as before.

THEOREM 1′. *The characteristic submanifold exists and is unique up to isotopy (in fact, ambient isotopy of manifolds with boundary pattern).*

EXAMPLE. If {connected components of ∂M} is a useful boundary pattern for M (that is, if ∂M is incompressible) the characteristic submanifold in the present sense coincides with the one defined previously, except that now the set of connected components of its intersection with ∂M has been designated as boundary pattern. Indeed the later fact is crucial in translating the notion of 'essential' from one setting to the other.

DEFINITION 3.3. Let $(M, \{F_i\})$ be any manifold with boundary pattern. Let N be a codimension zero submanifold of M such that $N \cap \partial M$ is a codimension zero submanifold of ∂M, in general position with respect to $\{F_i\}$. Then N is naturally endowed with what we refer to as its *proper boundary pattern*, given by

$$\bigcup_i \{\text{connected components of } V \cap F_i\}$$
$$\cup \{\text{connected components of } \mathrm{Cl}(\partial N \cap \mathrm{Int}(M))\}.$$

Note that the inclusion of N is not a map of manifolds with boundary patterns, in general.

Notation 3.4. Let $(M, \{F_i\})$ be a 3-manifold with boundary pattern, not necessarily complete. Assume the completed boundary pattern is useful. Then the characteristic submanifold V may be constructed, with respect to this completed boundary pattern. V may be endowed with its proper boundary pattern, with respect to $\{F_i\}$. It is this, V together with its proper boundary pattern, which by an abuse of language we will refer to as the *characteristic submanifold* of $(M, \{F_i\})$.

3.5. Similarly, $\mathrm{Cl}(M - V)$ may be endowed with its proper boundary pattern. It will, in general, have certain 'trivial components' of the type considered in Example 2.6; any such trivial component (with its completed boundary pattern) is isomorphic to either $S^1 \times S^1 \times I$, or $S^1 \times I \times I$, or $I \times I \times I$, respectively.

Throwing away the trivial components one obtains a manifold with boundary pattern, $(M^*, \{F_j^*\})$, say, which is referred to as being obtained from $(M, \{F_i\})$ by *splitting at its characteristic submanifold.*

LEMMA. *$(M^*, \{F_j^*\})$ is a manifold with useful boundary pattern. Furthermore it is simple in the sense that any component of its characteristic submanifold is contained in a neighborhood of one of the F_j^*, or of a component of* $\mathrm{Cl}(\partial M^* - \bigcup F_j^*)$.

REMARK. If $(M, \{F_i\})$ is simple then splitting at its characteristic submanifold does not change it, up to isomorphism.

3.6. Let $(M, \{F_i\})$ be a 3-manifold with useful boundary pattern. We consider incompressible surfaces S in M. We insist on considering only surfaces S not separating M (resp., ∂S not separating ∂M) provided there is at least one such surface. The latter of these is the case if M has at least one component that is neither closed nor a ball. We also insist that S be in general position with respect to $\{F_i\}$. A numerical function $c(S)$, the *complexity*, is defined by

$$c(S) = (\text{number of points } S \cap \textstyle\bigcup \partial F_i) + 5 \cdot (\text{first Betti number of } S).$$

Let $U(S)$, a regular neighborhood, and $\mathrm{Cl}(M - U(S))$ both be endowed with their proper boundary patterns. To the latter we refer as the manifold obtained by *splitting $(M, \{F_i\})$ at S.*

LEMMA. *Let S be such that $c(S)$ is minimal. Then the proper boundary patterns of $U(S)$ and $\mathrm{Cl}(M - U(S))$, respectively, are useful.*

From now on, the surfaces involved in a 3-manifold with boundary pattern will be dropped from the notation.

Let M_1 be a 3-manifold with boundary pattern, satisfying that the completed boundary pattern is useful. One forms M_2 by splitting M_1 at its characteristic submanifold, as in 3.5. In M_2 one picks some incompressible surface, nonseparating (etc.) if possible, of minimal complexity, and forms M_3 by splitting M_2 at S, as in 3.6. In general one forms M_{2i} by splitting M_{2i-1} at the characteristic submanifold, and M_{2i+1} by splitting M_{2i} at some S of minimal complexity. The process must stop after a finite number of steps (in the sense that any component of the manifold left over is some ball with boundary pattern); in fact, the argument of Haken [4] gives an explicit upper bound for this number. All the M_j in the sequence satisfy that the completed boundary pattern is useful; all the M_{2i} are simple.

DEFINITION 3.7. Any sequence $M_1, M_2, M_3, \cdots$ obtained in this way is called a *great hierarchy* for M_1.

It is by induction on a great hierarchy that one proves Theorems 2 and 3. In the inductive step one uses the following notion about 2-manifolds.

DEFINITION AND LEMMA. *Let F be a 2-manifold with complete boundary pattern, and let F_1 and F_2 be essential 2-submanifolds of F. Then there exists an essential 2-submanifold F_0 of F, unique up to isotopy, with the following properties*:

(i) *F_0 can be isotoped into both of F_1 and F_2;*

(ii) *any essential curve or arc in F, possibly singular, that can be deformed into both F_1 and F_2, respectively, can also be deformed into F_0;*

(iii) *no proper subcollection of components of F_0 has property* (ii).

F_0 is called the virtual intersection of F_1 and F_2.

One uses it in the following way.

3.8. Let $M_1, M_2, M_3, \cdots$ be a great hierarchy. Let M be the component of M_{2i} that contains the surface S, let $M' = \mathrm{Cl}(M - U(S))$ and V' the characteristic submanifold of M'. One desires to construct a submanifold P of M that consists of 'nice' pieces of components of V', fitting properly together across $U(S)$, and so that P is as large as possible. Here is a rough sketch of the construction. The components of V' that are Seifert fibre spaces can meet $\partial U(S)$ in a very special way only (in a neighborhood of a system of curves); for simplicity we assume there are none. For simplicity we assume further that any component of V' is a trivial line bundle rather than a nontrivial one (since S has minimal complexity this is in fact automatically true if S is nonseparating). Identify $U(S)$ with $S \times I$. Then by inductive application of the preceding lemma one finds that there is a largest sub-line-bundle V'' of V' satisfying that the virtual intersection, in S, of $V'' \cap (S \times 0)$ and $V'' \cap (S \times 1)$, is represented by these two surfaces themselves. Thus the components of V'' can now be fitted together, across $U(S)$, to form P. It is immediate from the construction that any component of P fibres over a 1-manifold, with fibre a 2-manifold. But M was assumed simple. So looking at ∂P one sees that in fact only two outcomes of the construction are possible: Either P is 'trivial' (that is, contained in a neighborhood of some surfaces of the completed boundary pattern), or P is essentially all of M, and M fibres over S^1. The preceding process will be referred to as *combing* of V'. In general, without the special assumptions we made, the process is more complicated. But one can still conclude that either P is trivial, in the above sense, or that P is essentially all of M; and in the latter case M either fibres over S^1, or is a union of two twisted line bundles glued at S (and in particular, some 2-sheeted covering of M fibres over S^1).

THEOREM 2', *Let M_1 be a 3-manifold with boundary pattern, both complete and useful. Then any essential singular torus, annulus, or square in M_1 can be deformed into the characteristic submanifold.*

REMARK. This includes Theorem 2 as the special case where the boundary pattern equals the set of connected components of ∂M_1.

INDICATION OF PROOF. One uses induction on a great hierarchy $M_1, M_2, M_3, \cdots$. The induction beginning is with a ball with boundary pattern. But here any essential square (the only case that can occur!) can be deformed into a nonsingular essential square, and hence into the characteristic submanifold, in view of the universal property defining the latter. The inductive step from M_{2i} to M_{2i-1} is of similar calibre. Where one really has to prove something is the step from M_{2i+1} to M_{2i}. Recall that M_{2i+1} is obtained from M_{2i} by splitting at some incompressible surface S, of minimal complexity, and nonseparating if possible. One assumes, contrary to the assertion, that there is an essential singular torus, say, call it f, that can neither be deformed off S, nor into a nonsingular torus in M_{2i}, this being necessarily some surface of the boundary pattern since M_{2i} is simple. In view of the induction hypothesis one may assume that the image of f is contained in the union of the regular neighborhood $U(S)$ and the characteristic submanifold of M_{2i+1}. In fact, one can apply the process of 'combing' of 3.8, and finds $\mathrm{Im}(f)$ can be contained in the manifold P constructed by combing. Thus P is nontrivial. Thus the component of M_{2i} that contains $\mathrm{Im}(f)$ fibres over S^1 (or at least some 2-sheeted covering does).

In this special case one produces a special proof, mainly by reading Nielsen **[14]**. Thus M_{2i} is not simple, a contradiction.

THEOREM 3'. *Let f_1: $M_1 \to N_1$ be a homotopy equivalence of manifolds with boundary patterns. One assumes the completed boundary patterns are useful. Then f_1 has singular support in the characteristic submanifold.*

REMARK. This includes Theorem 3 as the special case of an empty boundary pattern.

INDICATION OF PROOF. One uses induction on a great hierarchy M_1, M_2, M_3, $\cdots$. Actually additional conditions of a technical nature have to be asked of the surfaces S involved in the great hierarchy; these will here be tacitly assumed. One first has to go down the hierarchy. This uses, inductively,

LEMMA. *f_1 can be deformed to be a homotopy equivalence both on the characteristic submanifold and its complement, either being endowed with the proper boundary pattern.*

LEMMA. *f_2: $M_2 \to N_2$ can be deformed to be a homotopy equivalence both on $U(S)$ and* $\mathrm{Cl}(M_2 - U(S))$, *either being endowed with the proper boundary pattern.*

Next one has to establish the induction beginning, i.e., prove the theorem in the case when the manifolds are balls with boundary patterns, and simple. This is not entirely trivial (the argument is a pleasant exercise on the Jordan curve theorem). And finally one has to work up the hierarchy again, i.e., assuming the theorem is true for f_{2i+2} (and hence also for f_{2i+1}), one must show that f_{2i}: $M_{2i} \to N_{2i}$ can be deformed into a homeomorphism. Again one invokes the notion of virtual intersection, on S, the surface such that $M_{2i+1} = \mathrm{Cl}(M_{2i} - U(S))$. This uses

LEMMA. *Let f: $F \to G$ be a homotopy equivalence of 2-manifolds with boundary patterns. Let F' denote F with its completed boundary pattern, and let F_1, F_2 be essential 2-submanifolds of F'. If F_1 and F_2 are singular supports for f then so is their virtual intersection.*

By the lemma, the homotopy equivalence $f_{2i}|S$ has a unique minimal singular support. The problem is to show this is empty.

One now applies the process of 'combing' of 3.8. The submanifold P produced must contain the minimal singular support of $f_{2i}|S$, by the lemma. As pointed out in 3.8, only two cases are possible for P since M_{2i} is simple. If P is trivial we are done. If not, P is essentially all of the component of M_{2i} that contains S, and this component is of a very special kind. So a special argument applies.

The burden of the proof is in establishing the above lemmas.

4. Classifications. We must insist here that 3-manifolds shall be given in some particular, and effective, way. Thus a 'compact 3-manifold' shall mean a finite simplicial complex of a particular kind **[19]**. These form a recursive set, i.e., they 'can be listed', M_1, M_2, $\cdots$; furthermore, given two, it is trivial to decide, by inspection, if these two are isomorphic.

The *homeomorphism problem* is to give a recipe which, given M, M', decides if M and M' are PL isomorphic, i.e., if M and M' have isomorphic subdivisions.

The *classification problem* is to give a list of representatives, one from each PL isomorphism class.

The two problems are equivalent, by what Haken calls the cheapological trick. Indeed, given a solution to the former, one constructs the list $M_1, M_2, \cdots$, above, but at each step one inquires if a PL isomorphic manifold has been listed before, and if so, drops the new one. Conversely, given a solution of the latter in terms of representatives $M_1, M_2, \cdots$, one from each PL isomorphism class, then, given M', one generates the list of manifolds PL isomorphic to M' (by subdivision and its converse) until one finds the place of M' in the list $M_1, M_2, \cdots$; similarly one finds the place of M and thus sees if it is PL isomorphic to M' or not.

To clarify the meaning of this kind of classification one best considers an example pointed out by H. Schubert. Namely a finite group may be specified by a particular kind of multiplication table. It can be decided by inspection if two such multiplication tables define isomorphic groups. Hence one can make a list of finite groups $G_1, G_2, \cdots$, one from each isomorphism class. Similarly one can make a sublist giving the simple groups, or sporadic simple groups, respectively. But neither is this procedure very practical for groups of order exceeding 10^{80}, say, nor is it surprising that, in this case, the procedure just fails to answer any interesting question whatsoever. For example, from the recipe how to make the list of sporadic simple groups one cannot infer, even theoretically, if this list is finite.

In contradistinction to this example there are, in the case of 3-manifolds, at least two reasons for attempting just this kind of classification. Firstly there is no classification whatsoever of finitely presented groups and hence (Markov **[13]**, cf. also **[1]**) of compact manifolds of dimension exceeding 3. Secondly, any kind of classification of a sufficiently large class of 3-manifolds must invariably be tied up with some interesting structure theory.

The idea of Haken's approach to the homeomorphism problem **[4]** may be put as follows. Let M be given and suppose it is actually 'known' that M has certain desirable properties, in particular that it is irreducible and sufficiently large (as there is no algorithm yet to decide if these properties hold, they must be 'known' in advance so that one may use them in constructions involved in algorithms).

Step 1. Consider the hierarchies of M which are as simple as possible, show there are only finitely many, up to isomorphism, and produce a list. (Note that one does not classify here up to isotopy as the number of hierarchies would then be infinite, in general. To have this stronger finiteness one needs to work with something like great hierarchies, cf. below.)

Step 2. Given N similarly, consider any pair of hierarchies of M and N, respectively, and decide, by inspection, if they match.

Haken has shown **[4]**, **[6]** that this idea can be made to work 'in general'. The exceptional phenomenon is very special indeed, but unfortunately one does not have much control on its occurrence. It concerns embedded submanifolds which fibre over S^1, have incompressible boundary, and do not contain any incompressible surfaces apart from those isotopic to a fibre or a component of the boundary.

To illustrate the phenomenon suppose that M itself fibres over S^1, that ∂M is incompressible (possibly empty), and that the only incompressible surfaces are those isotopic to the fibres or the boundary components, respectively. If M' is similar then any homeomorphism $M \to M'$ must be isotopic to a fibre preserving one. So, presenting M as the mapping torus of a homeomorphism $f\colon F \to F$

(where F is the fibre of M), and M' similarly, the problem to decide if M and M' are homeomorphic is equivalent to deciding the following problem.

Problem. Given (F, f) and (F', f'), does there exist a homeomorphism $h: F \to F'$ so that hf is isotopic to $f'h$?

That is, one wants a solution to the conjugacy problem in the group of isotopy classes of automorphisms of a compact 2-manifold. Actually, one needs the solution only in a special case, but the special assumption seems hard to use, in general.

This problem eluded solution for a long time until very recently G. Hemion solved it at least in the case of nonempty boundary [7]. The solution is as follows.

Let $\tilde{F} \to F$ be a universal covering for F, and $\Delta \subset \tilde{F}$ a *fundamental domain*, that is, an embedding of the 2-disk so that $\Delta \to F$ is surjective. It is convenient to assume that $\Delta \to F$ is particularly nice, but this is not really relevant. For any f : $F \to F$, and any lifting $\tilde{f}: \tilde{F} \to \tilde{F}$, there exist elements $g_1, \cdots, g_n$ of the covering translation group so that $\tilde{f}(\Delta) \subset \bigcup_j g_j(\Delta)$. The minimal number of such group elements is denoted $d_\Delta(f)$, the *diameter of f with respect to Δ*. It depends on $\Delta \to F$, but not on any other choices.

If f fixes a point x, and $\tilde{x} \subset \tilde{F}$ is the pre-image of x, then f is determined, up to isotopy, by the permutation of $\tilde{x}$ induced by some lifting $\tilde{f}$. This makes it clear that for any given n' there exist only finitely many f, up to isotopy, with $d_\Delta(f) \leqq n'$, and furthermore that there is an effective procedure (by trial and error, say) to construct them all (or at least, to construct a slightly larger set).

THEOREM 4 (HEMION). *Suppose $\partial F \neq \emptyset$. Let f, f': $F \to F$ be given. Suppose f has neither periodic arcs, up to homotopy, nor periodic curves not deformable into ∂F. Suppose h: $F \to F$ satisfies that hf is isotopic to $f'h$. Then for some integer m, and some h' isotopic to hf^m, it is true that*

$$d_\Delta(h') \leqq \$(d_\Delta(f), d_\Delta(f'))$$

where \$ is some explicitly given function of two variables.

In view of known results (implicitly used below) the general solution of the above problem follows from this theorem if $\partial F \neq \emptyset$.

The solution of the homeomorphism problem will now be described in a way that explicitly uses great hierarchies. As a bonus, additional results can be obtained on the classification of homeomorphisms. Let M be as before: It is (known to be) irreducible, and if it is closed it is also (known to be) sufficiently large. If $\partial M \neq \emptyset$ we assume M equipped with some boundary pattern $\{F_i\}$ which is both complete and useful (an algorithm of Haken can be used to check the latter). As before, S denotes an incompressible surface in M (nonseparating, etc., if such exist at all) with complexity $c(S)$ defined by

$$\left(\text{number of points } S \cap \bigcup \partial F_i\right) + 5 \cdot (\text{first Betti number of } S).$$

THEOREM 5 ([2], [3], [4]). (i) *There is an algorithm to construct the characteristic submanifold of M.*

(ii) *If M is simple, cf.* 3.5, *then there is only a finite number, up to ambient isotopy,* *of surfaces S of minimal complexity. There is an algorithm which produces one such surface from each isotopy class.*

REMARK. An algorithm of Haken can be used to decide if two incompressible surfaces are isotopic. Thus the final assertion of the theorem may be strengthened to say that precisely one S is produced from each isotopy class.

Let $(M_1, M_2, M_3, \cdots)$ and $(M'_1, M'_2, M'_3, \cdots)$ be great hierarchies, in the sense of 3.10. An *isomorphism* from one to the other is a sequence of PL isomorphisms $f_j: M_j \to M'_j$ such that

(i) for $j = 2i$, and the surfaces S and S' used in passage from $2i$ to $2i+1$, the surfaces $f_j(S)$ and S' in M'_j are ambient isotopic;

(ii) for any j, f_{j+1} is induced, up to isotopy, from f_j.

If $M_1 = M'_1$ and f_1 is the identity, the isomorphism will be called an *isotopy*. In view of the fact that the lengths of great hierarchies of M are uniformly bounded above [4], Theorem 5 thus gives

COROLLARY. *There is only a finite number, up to isotopy, of great hierarchies of M. There is an algorithm which produces precisely one from each class.*

Let $\mathscr{H}(M, N)$ denote the set of isotopy classes of homeomorphisms from M to N, and $\mathscr{H}(M)$ the group $\mathscr{H}(M, M)$. If $h: M \to M$ is a homeomorphism, we will say h has *support in* $M' \subset M$ if there exists h' isotopic to h so that $h'|M - M'$ is the identity; these form the subgroup $\mathscr{H}_{M'}(M)$ of $\mathscr{H}(M)$. For example, a Dehn twist is an automorphism of a 2-manifold with support in the neighborhood of an embedded circle. Similarly, an automorphism of a 3-manifold may be called a *Dehn twist* if it has support in the neighborhood of an essential annulus or torus.

We denote by $bp(M)$ the set of surfaces involved in the boundary pattern of M, and by $hier(M)$ the set of isotopy classes of 'oriented' great hierarchies (that is, M, and each of the surfaces S involved, is endowed with some orientation).

THEOREM 6. *Let M_2 be connected and simple. Suppose M_2 does not fibre over S^1, nor, if it is closed, that it is the union of two twisted line bundles. Then*

$$\mathscr{H}(M_2) \to \mathrm{Aut}(bp(M_2)) \times \mathrm{Aut}(hier(M_2))$$

is injective. If N_2 is similar then the image of $\mathscr{H}(M_2, N_2)$ in

$$\mathrm{Hom}(bp(M_2), bp(N_2)) \times \mathrm{Hom}(hier(M_2), hier(N_2))$$

is a computable set; in particular it can be decided whether or not $\mathscr{H}(M_2, N_2)$ is empty.

INDICATION OF PROOF. Let $f_2: M_2 \to M_2$ be such that $bp(f_2)$ is the identity, and $hier(f_2)$ has a fixed point. By induction on the length of the great hierarchy, f_2 must then be isotopic to the identity, as follows. Let $M_2, M_3, \cdots$ represent this fixed point, where $M_3 = \mathrm{Cl}(M_2 - U(S))$, etc. It may be assumed that $f_2(S) = S$, $f_2(M_3) = M_3$, $f_2(M_4) = M_4$, and, by the inductive hypothesis, that $f_2|M_4$ is the identity. It is clear, more or less, that it suffices to prove $f_2|S$ is isotopic to the identity. One uses

LEMMA. *Let $f: S \to S$ be an automorphism of a 2-manifold with complete bound-*

ary pattern. Let F_1, F_2 be essential 2-submanifolds of S. If F_1 and F_2 are supports for f then so is their virtual intersection.

One may thus apply the process of combing the characteristic submanifold of M_3, cf. 3.8, and the submanifold P produced will contain the minimal support of $f_2|S$, by the lemma. But P must be trivial, in view of the hypothesis that M_2 does not fibre, etc.

The second part of the theorem follows since the combing process is really a constructive method, and one can thus compare, by inspection, the data in M_2 and N_2, respectively.

THEOREM 7. *Let M_2 be connected and simple. Suppose that $\partial M_2 \neq \emptyset$ and that M_2 fibres over S^1. Let N_2 be similar. Then $\mathscr{H}(M_2, N_2)$ is a finite computable set.*

PROOF. By Haken, there are, up to isotopy, only finitely many fibrations of M_2 whose fibre is of minimal complexity, and all of these can be constructed. It thus suffices to consider only fibre preserving homeomorphisms. Let M_2 be the mapping torus of a homeomorphism f: $F \to F$. Then f does not have periodic arcs or curves, up to homotopy, which cannot be deformed into ∂F because otherwise M_2 would not be simple (by Theorem 2, or really Nielsen **[14]**). The assertion is thus immediate from Theorem 4.

In the following we let M be a connected 3-manifold as specified earlier, but we exclude these two cases:

(i) M is closed and simple and fibres over S^1;

(ii) M is closed and simple and is the union of two line bundles.

Note that it can be checked by Haken's algorithm if M is one of these. $\mathscr{H}_V(M)$ denotes the normal subgroup of $\mathscr{H}(M)$ of homeomorphism classes with support in V, the characteristic submanifold of M.

THEOREM 8. *If M is as just specified, and N similarly, then $\mathscr{H}_V(M)\backslash\mathscr{H}(M, N)$ is a finite computable set.*

INDICATION OF PROOF. Let M' be obtained from M by splitting at the characteristic submanifold, and N' similarly. The hypothesis about M implies that each component of M' satisfies the hypothesis of either Theorem 6 or 7. Thus $\mathscr{H}(M', N')$ is a finite computable set. By definition, M' was obtained from $\mathrm{Cl}(M - V)$ by discarding certain trivial components. Thus we must now add V to M', and those trivial components. For example, one component of $\mathrm{Cl}(M - M')$ could be a graph manifold in the sense of **[22]**, cf. Example 2.6 above. To proceed one can, e.g., use explicit knowledge about such special manifolds and their homeomorphisms.

COROLLARY. *Let M denote a compact orientable irreducible* 3*-manifold with nonempty incompressible boundary.*

1. *These M can be classified* (*by Theorem* 8).
2. *The M' homotopy equivalent to M can be classified* (*by Theorem* 3).
3. *These $\pi_1 M$ can be classified* (*by* 1 *and* 2).
4. *Knot groups can be classified* (*by* 3).
5. *Knots can be classified* (*by* 1, *and inclusion of a meridian in the data*).

COROLLARY. *If M is as in Theorem* 8, *$\mathscr{H}(M)/\mathscr{H}_V(M)$ is a finite computable group. In particular if M is simple, $\mathscr{H}(M)$ is a finite computable group and hence so is the group of automorphism classes of $\pi_1 M$, by Theorem* 3.

The following remarks show that $\mathscr{H}_V(M)$ is also computable though in general not finite. Let N be a Seifert fibre space with decomposition surface B. Let G denote the group of fibre preserving homeomorphisms modulo fibre preserving isotopy. It was indicated in **[24]** that, in general, the map $G \to \mathscr{H}(N)$ is an isomorphism (the exceptions are (i) those of Example 1.4, (ii) the Seifert fibre spaces of 2.2, plus alternative Seifert fiberings of these, (iii) finitely many others, e.g., the 3-torus). Let the chain of subgroups $G_3 \subset G_2 \subset G_1 \subset G$ be defined by

(G_1) h is orientation preserving,

(G_2) h maps each exceptional fibre to itself, by an orientation preserving map,

(G_3) h maps each fibre to itself, by an orientation preserving map.

Then G_1/G_2 is always finite. In the case where B is orientable, it is a product of permutation groups (namely those exceptional fibres may be permuted that can be permuted) and possibly a Z_2. In the other case, the structure is slightly more complicated in that each exceptional fibre may be flipped individually.

$G_2/G_3 \xrightarrow{\approx} \mathscr{H}(B)$ where B is to be considered as a pointed 2-manifold, pointed by the exceptional fibres. Generators for this group have long been known, in particular $\mathscr{H}(B)$ has a subgroup of finite index which is generated by Dehn twists. A system of relators has recently been obtained by Hatcher and Thurston (not yet published), in particular it is now known that $\mathscr{H}(B)$ is finitely presented.

$G_3 \xrightarrow{\approx} H_1(B, \partial B)$ (with two exceptions; cf. below). A cocycle in the dual group $H_1(B; Z)$ is represented by a section in a certain S^1-bundle over B; the section can be interpreted to measure how the associated element of G_3 rotates the nonexceptional fibres. If k is a nonsingular arc in B not containing an exceptional point then the element of $H^1(B, \partial B)$ represented by k corresponds to a primitive Dehn twist along the annulus over k; similarly, if k is a closed curve, it corresponds to a primitive Dehn twist at a torus, in fibre direction.—The exceptions are given by the S^1-bundles over the torus and the Klein bottle, respectively. The exceptional phenomenon is that in these cases there exist nontrivial isotopies which slide the fibres around (such exceptional isotopies exist in three more cases, but here they do not do anything). The effect is that $H_1(B, \partial B)$ has to be replaced by the quotient group

$$H_1(B, \partial B) \otimes Z/nZ$$

where n is the Euler number in the case where B is the torus, and twice the Euler number in the case of the Klein bottle.

Similar but simpler considerations apply to line bundles. Putting together these considerations involving the work of Haken, Hemion, Johannson, Hatcher-Thurston, one has

COROLLARY. *Let M be as in Theorem* 8, *but in addition exclude the case where M is a Seifert fibre space over S^2 with precisely three exceptional fibres* (*that is, one must explicitly exclude now only those which are sufficiently large*). *Then $\mathscr{H}(M)$ is a finitely presented computable group.*

REMARK. In the case additionally excluded in the corollary, it is true that $\mathscr{H}(M)$ is isomorphic to the group of automorphism classes of $\pi_1 M$. Therefore results of Zieschang [29] show that the corollary extends to this case.

COROLLARY. *Let M be as in the preceding corollary. Then the normal subgroup of $\mathscr{H}(M)$ generated by Dehn twists at essential annuli and tori is of finite index.*

REFERENCES

1. W. W. Boone, W. Haken and V. Poénaru, *On recursively unsolvable problems in topology and their classification*, Contributions to Math. Logic, North-Holland, Amsterdam, 1968, pp. 37–74. MR **41** #7695.

2. W. Haken, *Theorie der Normalflächen*, Acta Math. **105** (1961), 245–375. MR **25** #4519A.

3. ———, *Ein Verfahren zur Aufspaltung einer 3-Mannigfaltigkeit in irreduzible 3-Mannigfaltigkeiten*, Math. Z. **76** (1961), 427–467. MR **25** #4519C.

4. ———, *Über das Homöomorphieproblem der 3-Mannigfaltigkeiten*. I, Math. Z. **80** (1962), 89–120. MR **28** #3410.

5. ———, *Some results on surfaces in 3-manifolds*, Studies in Modern Topology, MAA Studies in Math., vol. 5, Prentice-Hall, Englewood Cliffs, N. J., pp. 39–98. MR **36** #7118.

6. ———, *Connections between topological and group theoretical decision problems*, Word Problems, North-Holland, Amsterdam, 1973, pp. 427–441.

7. G. Hemion, *On the classification of homeomorphisms of 2-manifolds and the classification of 3-manifolds*, Univ. Bielefeld (In prep.).

8. K. Johannson, *On essential embeddings of annuli and tori in irreducible 3-manifolds which are sufficiently large*, Univ. Bielefeld, Aug. 1974 (preliminary version of [10]).

9. ———, *Equivalences d'homotopie des variétés de dimension* 3, C. R. Acad. Sci. Paris Sér. A **281** (1975), 1009–1010. MR **52** #11918.

10. ———, *Homotopy equivalences of knot spaces*, Univ. Bielefeld, Feb. 1976.

11. ———, *Homotopy equivalences of 3-manifolds with boundary*, Univ. Bielefeld, Aug. 1976.

12. H. Kneser, *Geschlossene Flächen in dreidimensionalen Mannigfaltigkeiten*, Jber. Deutsch. Math.-Verein. **38** (1929), 248–260.

13. A. A. Markov, *Insolubility of the problem of homeomorphy*, Proc. Internat. Congress Math. (1958). Cambridge Univ. Press, London and New York, 1960, pp. 300–306. (Russian) MR **22** #5962.

14. J. Nielsen, *Untersuchungen zur Topologie der geschlossenen zweiseitigen Flächen*. II, Acta Math. **53** (1929), 1–76.

15. ———, *Abbildungsklassen endlicher Ordnung*, Acta Math. **75** (1943), 23–115. MR **7**, 137.

16. C. D. Papakyriakopoulos, *On Dehn's lemma and the aspherity of knots*, Ann. of Math. (2) **66** (1957), 1–26. MR **19**, 761.

17. ———, *On solid tori*, Proc. London Math. Soc. (3) **7** (1957), 281–299. MR **19**, 441.

18. H. Seifert, *Topologie dreidimensionaler gefaserter Räume*, Acta Math. **60** (1933), 147–238.

19. H. Seifert and W. Threlfall, *Lehrbuch der Topologie*, Teubner, Leipzig, 1934.

20. J. R. Stallings, *On the loop theorem*, Ann. of Math. (2) **72** (1960), 12–19. MR **22** #12526.

21. ———, *On fibering certain 3-manifolds*, Topology of 3-manifolds and Related Topics, Prentice-Hall, Englewood Cliffs, N. J., 1962, pp. 95–100. MR **28** #1600.

22. F. Waldhausen, *Eine Klasse von 3-dimensionalen Mannigfaltigkeiten*. I, II, Invent. Math. **3** (1967) 308–333; ibid. **4** (1967), 87–117. MR **38** #3880.

23. ———, *Gruppen mit Zentrum und 3-dimensionale Mannigfaltigkeiten*, Topology **6** (1967), 505–517. MR **38** #5223.

24. ———, *On irreducible 3-manifolds which are sufficiently large*, Ann. of Math. (2) **87** (1968), 56–88. MR **36** #7146.

25. ———, *The word problem in fundamental groups of sufficiently large 3-manifolds*, Ann. of Math. (2) **88** (1968), 272–280. MR **39** #2167.

26. F. Waldhausen, *On the determination of some bounded 3-manifolds by their fundamental groups alone*, Proc. Sympos. on Topology and its Applications, Herceg-Novi, 1968 (Beograd, 1969), 331–332. MR **42** #2416.

27. ———, *Some problems on 3-manifolds*, these PROCEEDINGS, part 2, pp. 313–322.

28. J. H. C. Whitehead, *On 2-spheres in 3-manifolds*, Bull. Amer. Math. Soc. **64** (1958), 161–166. MR **21** #2241.

29. H. Zieschang, E. Vogt and H. D. Coldewey, *Flächen und ebene diskontinuierliche Gruppen*, Lecture Notes in Math., vol. 122, Springer-Verlag, Berlin and New York, 1970. MR **41** #6986.

UNIVERSITÄT BIELEFELD